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Dependence of α in Peak Norms and Best Peak Norms Approximation

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Let $C[0, 1]$ be the space of all continuous functions defined on $[0, 1]$ and U be

denotes the Lebesgue measure. We say $p \in U$ is a best α-norm approximant to f from U if $D_\alpha(f) = \|f - p\|_\alpha = \inf\{\|f-u\|_\alpha \mid u \in U\}$. In this paper we shall study $\|f\|_\alpha$, $D_\alpha(f)$ and $P_\alpha(f) = \{p \in U \mid \|f-p\|_\alpha = D_\alpha(f)\}$ as functions of α for fixed f. We shall show their continuous dependence on α and differentiability with respect to α.

Key Words: best approximation; peak norm; α-norm; continuity; differentiability.

1. INTRODUCTION

Let $C[0, 1]$ be the space of all continuous functions defined on $[0, 1]$ and U be an n dimensional subspace of $C[0, 1]$. For $0 < \alpha \leqslant 1$, the peak norm, or α-norm is defined by

$$\|f\|_\alpha = \frac{1}{\alpha} \sup \left\{ \int_A |f| \, d\mu \mid \mu(A) = \alpha, \, A \subset [0, 1] \right\},$$

where μ denotes the Lebesgue measure. We say $p_\alpha \in U$ is a best α-norm approximant to f from U if

$$D_\alpha(f) = \|f - p_\alpha\|_\alpha = \inf\{\|f - u\|_\alpha \mid u \in U\}.$$

α-norm and best α-norm approximation were introduced and discussed in [4], and also in [6]. α-norms serve as a bridge between the classical uniform norm and L^1 norm, because it is L^1 norm when $\alpha = 1$ and $\lim_{\alpha \to 0} \|f\|_\alpha = \|f\| = \max_{0 \leqslant x \leqslant 1} |f(x)|$, the uniform norm of f. Best α-norm approximation has both an L^1-type characterization theorem and alternating

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property [4]. A sufficient condition for the uniqueness of best α-norm approximation is given in [6]: "U is an A-space and $\mu(Z(u)) < \alpha$ for any $0 \not\equiv u \in U$, where $Z(u) = \{x \mid u(x) = 0\}$." This condition becomes that U is an A-space when $\alpha = 1$ and U is a Chebyshev space on $(0, 1)$ when $\alpha \to 0$. Recall that an A-space guarantees the uniqueness of the best L^1 approximation and a Chebyshev space guarantees the uniqueness of the best uniform approximation. Recently the peak L^p norms are studied in [5].

In this paper we shall study $\|f\|_\alpha$, $D_\alpha(f)$ and $P_\alpha(f) = \{p \in U \mid \|f - p\|_\alpha = D_\alpha(f)\}$ as functions of α for a fixed function f. We shall show their continuous dependence on α and differentiability with respect to α.

2. CONTINUITY IN α

We begin with stating some known results:

THEOREM 2.1 [4, 6]. *Let $f \in C[0, 1]$, U, $D_\alpha(f)$ and $P_\alpha(f)$ be defined as above. Then*

(1) *If $0 < \beta < \alpha \leqslant 1$, then*

$$\|f\|_\alpha \leqslant \|f\|_\beta \leqslant \frac{\alpha}{\beta} \|f\|_\alpha$$

and

$$D_\alpha(f) \leqslant D_\beta(f) \leqslant \frac{\alpha}{\beta} D_\alpha(f).$$

(2) *If U is an A-space and $\mu(Z(u)) < \alpha$ for any $0 \not\equiv u \in U$, then there exists a $\delta > 0$ such that the best β-norm approximation of f is unique for all $\beta > \alpha - \delta$, which is denoted by $p_\beta(f)$, and*

$$\lim_{\beta \to \alpha} p_\beta(f) = p_\alpha(f), \qquad 0 < \alpha < 1,$$

and

$$\lim_{\beta \to 1^-} p_\beta(f) = p_1(f).$$

(3) *If U is a Cheyshev space, then*

$$\lim_{\beta \to 0^+} p_\beta(f) = p_0(f),$$

where $p_0(f)$ denotes the unique best uniform norm approximant of f.

We need some more notations.

Let $A_\alpha(f)$ denote any α-norm norming set of f; i.e., $\mu(A_\alpha(f))=\alpha$ and

$$\frac{1}{\alpha}\int_{A_\alpha(f)}|f|=\|f\|_\alpha.$$

In what follows, we always choose $A_\alpha(f)\subset A_\beta(f)$ whenever $\alpha\leqslant\beta$.

Let

$$h_\alpha(f)=\inf\{h\mid\mu\{x\in[0,1]\mid|f(x)|\geqslant h\}\leqslant\alpha\}.$$

It is worth noting that for any norming set $A_\alpha(f)$

$$\{x\mid|f(x)|>h_\alpha(f)\}\subset A_\alpha(f)\subset\{x\mid|f(x)|\geqslant h_\alpha(f)\}.$$

Let

$$E(f)=\{x\mid|f(x)|=\|f\|\},$$

where $\|\cdot\|$ denotes the uniform norm.

Also, for simplicity, $\alpha\to 0(1)$ means $\alpha\to 0^+(1^-)$, and $f'_+(x)$ is considered only for $0\leqslant x<1$ and $f'_-(x)$ is considered only for $0<x\leqslant 1$.

THEOREM 2.2. *Let $P_\alpha=P_\alpha(f)$. For $0\leqslant\alpha\leqslant 1$, we have*

$$\lim_{\beta\to\alpha}\sup_{p\in P_\beta}\inf_{q\in P_\alpha}\{\|p-q\|\}=0.$$

Proof. Suppose that the above limit does not go to 0, then there exist α_k, $k=1,2,\ldots$ with $|\alpha_k-\alpha|\leqslant\frac{1}{k}$ and $p_k\in P_{\alpha_k}$ such that

$$\inf_{q\in P_\alpha}\{\|q-p_k\|\}>\frac{1}{k}. \tag{1}$$

Since $\{p_k\}$ is bounded, by compactness of a closed bounded set in a finite dimensional space, there is a subsequence $\{p_{k_j}\}$ converging to p. Then, for $0<\alpha\leqslant 1$,

$$\begin{aligned}\|f-p\|_\alpha&=\lim_{j\to\infty}\|f-p_{k_j}\|_\alpha\leqslant\lim_{j\to\infty}\max\left\{1,\frac{\alpha}{\alpha_{k_j}}\right\}\|f-p_{k_j}\|_{\alpha_{k_j}}\\&=\lim_{j\to\infty}D_{\alpha_{k_j}}(f)=D_\alpha(f).\end{aligned}$$

The last inequality follows from Theorem 2.1. This means that $p\in P_\alpha$ and it contradicts (1).

For $\alpha = 0$ and $p \notin P_0$,

$$\|f - p\| > \|f - q\| = D_0(f)$$

and there exist $x_0 \in [0, 1]$ and $\varepsilon_0 > 0$ such that

$$|f(x_0) - p(x_0)| > \|f - q\| + 3\varepsilon_0.$$

By the continuity of f and the fact $\lim_{j \to \infty} \|p_{k_j} - p\| = 0$, there exist $m > 0$ such that for $j > m$ and $|x - x_0| < \frac{1}{m}$.

$$|f(x) - p_{k_j}(x)| > \|f - q\| + \varepsilon_0 \qquad \text{and} \qquad \alpha_{k_j} < \frac{1}{m}$$

and then

$$\|f - p_{k_j}\|_{\alpha_{k_j}} \geqslant \|f - q\| + \varepsilon_0 > \|f - q\|_{\alpha_{k_j}}.$$

This contradicts that $p_{k_j} \in P_{\alpha_{k_j}}$.

The next two lemmas show the continuity of $h_\alpha(f - p)$ with $p \in P_\alpha(f)$. These results will be used in proving the differentiability of $D_\alpha(f)$.

Lemma 2.3. *If $\beta < \alpha$, then*

$$\sup_{q \in P_\alpha(f)} \{h_\alpha(f - q)\} \leqslant \inf_{p \in P_\beta(f)} \{h_\beta(f - p)\}.$$

Proof. For any $p_\alpha \in P_\alpha$ and $p_\beta \in P_\beta$ with $\beta < \alpha$,

$$\begin{aligned}
\int_{A_\alpha(f - p_\alpha)} |f - p_\alpha| = \alpha D_\alpha(f) \leqslant \int_{A_\alpha(f - p_\beta)} |f - p_\beta| \\
&= \int_{A_\beta(f - p_\beta)} |f - p_\beta| + \int_{A_\alpha(f - p_\beta) - A_\beta(f - p_\beta)} |f - p_\beta| \\
&\leqslant \beta D_\beta(f) + (\alpha - \beta)\, h_\beta(f - p_\beta) \\
&\leqslant \int_{A_\beta(f - p_\alpha)} |f - p_\alpha| + (\alpha - \beta)\, h_\beta(f - p_\beta) \\
&= \int_{A_\beta(f - p_\alpha)} |f - p_\alpha| + (\alpha - \beta)\, h_\alpha(f - p_\alpha) \\
&\quad - (\alpha - \beta)\, h_\alpha(f - p_\alpha) + (\alpha - \beta)\, h_\beta(f - p_\beta)
\end{aligned}$$

$$\leqslant \int_{A_\beta(f-p_\alpha)} |f-p_\alpha| + \int_{A_\alpha(f-p_\alpha)-A_\beta(f-p_\alpha)} |f-p_\alpha|$$

$$-(\alpha-\beta)\, h_\alpha(f-p_\alpha) + (\alpha-\beta)\, h_\beta(f-p_\beta)$$

$$= \int_{A_\alpha(f-p_\alpha)} |f-p_\alpha| + (\alpha-\beta)(h_\beta(f-p_\beta) - h_\alpha(f-p_\alpha)).$$

Thus $h_\beta(f-p_\beta) - h_\alpha(f-p_\alpha) \geqslant 0$.

LEMMA 2.4. *Let* $0 \leqslant \alpha \leqslant 1$ *and* $f \in C[0,1]$. *Then*

$$\lim_{\beta \to \alpha^+} \sup_{q \in P_\beta(f)} \{|h_\beta(f-q) - \inf_{p \in P_\alpha(f)} \{h_\alpha(f-p)\}|\} = 0 \tag{2}$$

and

$$\lim_{\beta \to \alpha^-} \sup_{q \in P_\beta(f)} \{|h_\beta(f-q) - \sup_{p \in P_\alpha(f)} \{h_\alpha(f-p)\}|\} = 0. \tag{3}$$

Proof. By Lemma 2.3, for $\beta > \alpha$,

$$\sup_{q \in P_\beta(f)} \{|h_\beta(f-q) - \inf_{p \in P_\alpha(f)} \{h_\alpha(f-p)\}|\}$$
$$= \sup_{q \in P_\beta(f)} \{\inf_{p \in P_\alpha(f)} \{|h_\beta(f-q) - h_\alpha(f-p)|\}\}$$

and for $\beta < \alpha$,

$$\sup_{q \in P_\beta(f)} \{|h_\beta(f-q) - \sup_{p \in P_\alpha(f)} \{h_\alpha(f-p)\}|\}$$
$$= \sup_{q \in P_\beta(f)} \{\inf_{p \in P_\alpha(f)} \{|h_\beta(f-q) - h_\alpha(f-p)|\}\},$$

and

$$\sup_{q \in P_\beta(f)} \{\inf_{p \in P_\alpha(f)} \{|h_\beta(f-q) - h_\alpha(f-p)|\}\}$$
$$\leqslant \sup_{q \in P_\beta(f)} \{\inf_{p \in P_\alpha(f)} \{|h_\beta(f-q) - h_\beta(f-p)|\}\}$$
$$+ \sup_{p \in P_\alpha(f)} \{|h_\beta(f-p) - h_\alpha(f-p)|\}.$$

By Theorem 2.2, the first term of the above expression goes to 0 as $\beta \to \alpha$. Since $P_\alpha(f)$ is a compact set and all functions in this set are continuous, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|p_\alpha(x) - p_\alpha(y)| < \varepsilon$ for any x, $y \in [0,1]$ with $|x-y| < \delta$ and any $p_\alpha \in P_\alpha(f)$. Thus the second term of the above expression goes to 0 too as $\beta \to \alpha$.

Thus,

$$\lim_{\beta \to \alpha} \sup_{q \in P_\beta(f)} \left\{ \inf_{p \in P_\alpha(f)} \{ |h_\beta(f-q) - h_\alpha(f-p)| \} \right\} = 0.$$

COROLLARY 2.5. *Let* $0 \leqslant \alpha \leqslant 1$ *and* $f \in C[0, 1]$. *If* $h_\alpha(f-p)$ *have the same value for all* $p \in P_\alpha(f)$, *then*

$$\lim_{\beta \to \alpha} \sup_{q \in P_\beta(f)} |h_\beta(f-q) - h_\alpha(f-p)| = 0.$$

When the best α-norm approximation is unique, i.e. $P_\alpha(f)$ is singleton, then $h_\alpha(f-p)$ of course has one value. However, if $P_\alpha(f)$ is not singleton, $h_\alpha(f-p)$ may be different for different $p \in P_\alpha(f)$. See Example 2 in the next section.

3. DIFFERENTIABILITY IN α

THEOREM 3.1. *Let* $f \in C[0, 1]$. *Then, for* $0 < \alpha \leqslant 1$, $\|f\|_\alpha$ *is differentiable with respect to* α, *and*

$$\frac{d}{d\alpha} \|f\|_\alpha = \frac{h_\alpha(f) - \|f\|_\alpha}{\alpha} \leqslant 0.$$

Proof. Choose norming sets $A_\alpha(f)$ and $A_\beta(f)$ such that $A_\alpha(f) \subset A_\beta(f)$ if $\alpha < \beta$, and $A_\beta(f) \subset A_\alpha(f)$ if $\beta < \alpha$. Let $A \Delta B = (A - B) \cup (B - A)$. Then

$$\begin{aligned}
&\frac{\|f\|_\beta - \|f\|_\alpha}{\beta - \alpha} \\
&= \frac{(1/\beta) \int_{A_\beta(f)} |f| - (1/\alpha) \int_{A_\alpha(f)} |f|}{\beta - \alpha} \\
&= \frac{(1/\beta - 1/\alpha) \int_{A_\alpha(f) \cup A_\beta(f)} |f| + \operatorname{sgn}\{\beta - \alpha\} \min\{1/\beta, 1/\alpha\} \int_{A_\alpha(f) \Delta A_\beta(f)} |f|}{\beta - \alpha} \\
&= -\frac{1}{\beta\alpha} \int_{A_\alpha(f) \cap A_\beta(f)} |f| + \frac{\min\{1/\beta, 1/\alpha\}}{|\beta - \alpha|} \\
&\quad \times \left[\int_{A_\alpha(f) \Delta A_\beta(f)} h_\alpha(f) + \int_{A_\alpha(f) \Delta A_\beta(f)} (|f(x)| - h_\alpha(f)) \right].
\end{aligned}$$

Since $A_\alpha(f)\cap A_\beta(f)=A_\alpha(f)$ when $\alpha\leqslant\beta(=A_\beta(f)$ when $\alpha\geqslant\beta)$, $\mu(A_\alpha(f)\,\Delta A_\beta(f))=|\beta-\alpha|$ and $\lim_{\beta\to\alpha}\|\,|f(x)|-h_\alpha(f)\|_{A_\alpha(f)\,\Delta A_\beta(f)}=0$, we have

$$\lim_{\beta\to\alpha}\frac{\|f\|_\beta-\|f\|_\alpha}{\beta-\alpha}=-\frac{1}{\alpha^2}\int_{A_\alpha(f)}|f|+\frac{1}{\alpha}h_\alpha(f)=\frac{h_\alpha(f)-\|f\|_\alpha}{\alpha}\leqslant 0.$$

The last inequality follows from the fact $h_\alpha(f)\leqslant\|f\|_\alpha$.

THEOREM 3.2. *Let $f\in C[0,1]$, then*

$$0\geqslant\limsup_{\alpha\to 0}\frac{\|f\|_\alpha-\|f\|}{\alpha}\geqslant\liminf_{\alpha\to 0}\frac{\|f\|_\alpha-\|f\|}{\alpha}$$

$$\geqslant-\left(\inf_{x\in E(f)}\left\{\limsup_{\substack{h\to 0,\,h>0\\ x+h\in[0,1]}}\left|\frac{f(x+h)-f(x)}{h}\right|(x\neq 1),\right.\right.$$

$$\left.\left.\limsup_{\substack{h\to 0,\,h<0\\ x+h\in[0,1]}}\left|\frac{f(x+h)-f(x)}{h}\right|(x\neq 0)\right\}\right).$$

Proof. Since $\|f\|_\alpha\leqslant\|f\|$, the first two inequalities are obvious. For any $x_0\in E(f)$, let

$$\limsup_{\substack{h\to 0,\,h>0\\ x_0+h\in[0,1]}}\left|\frac{f(x_0+h)-f(x_0)}{h}\right|=\lambda.$$

Then for any $\varepsilon>0$ there exists a δ such that for $0<h<\delta$,

$$\left|\frac{f(x_0+h)-f(x_0)}{h}\right|<\lambda+\varepsilon,$$

or

$$\frac{|f(x_0+h)|-|f(x_0)|}{h}>-\lambda-\varepsilon.$$

Then, for $\alpha<\delta$,

$$\frac{\|f\|_\alpha-\|f\|}{\alpha}\geqslant\frac{1}{\alpha^2}\int_{[x_0,\,x_0+\alpha]}(|f(x)|-\|f\|)=\frac{1}{\alpha^2}\int_{[x_0,\,x_0+\alpha]}(|f(x)|-|f(x_0)|)$$

$$\geqslant\frac{1}{\alpha}\int_{[x_0,\,x_0+\alpha]}\frac{|f(x)|-|f(x_0)|}{x-x_0}>-\lambda-\varepsilon.$$

This shows

$$\liminf_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} \geqslant - \limsup_{\substack{h \to 0,\, h > 0 \\ x+h \in [0,1]}} \left| \frac{f(x+h) - f(x)}{h} \right|.$$

The proof of

$$\liminf_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} \geqslant - \limsup_{\substack{h \to 0,\, h < 0 \\ x+h \in [0,1]}} \left| \frac{f(x+h) - f(x)}{h} \right|$$

is similar. Combining these inequalities proves the theorem.

COROLLARY 3.3. *Let $f \in C[0,1]$.*

(1) *If $\{x \mid x \in E(f)$ and either $f'_+(x)$ or $f'_-(x)$ exists$\} \neq \phi$, then*

$$0 \geqslant \limsup_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} \geqslant \liminf_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha}$$

$$\geqslant -\big(\inf_{x \in E(f)} \{ |f'_-(x)|, |f'_+(x)| \} \big).$$

(2) *If $\inf_{x \in E(f)} \{ |f'_-(x)|, |f'_+(x)| \} = 0$, then*

$$\lim_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} = 0.$$

THEOREM 3.4. *Let $f \in C[0,1]$. If both $f'_+(x)$ and $f'_-(x)$ exist or is $\pm\infty$ for any $x \in E(f)$, then*

$$\lim_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} = \begin{cases} -\dfrac{1}{2 \sum_{x \in E(f)} \left(\dfrac{1}{|f'_+(x)|} + \dfrac{1}{|f'_-(x)|} \right)} & 0 < \min_{x \in E(f)} \{ |f'_+(x)|, |f'_-(x)| \} < \infty, \\ 0 & \inf_{x \in E(f)} \{ |f'_+(x)|, |f'_-(x)| \} = 0, \\ -\infty & \min_{x \in E(f)} \{ |f'_+(x)|, |f'_-(x)| \} = \infty, \end{cases}$$

where $f'_-(0)$ is not considered if $0 \in E(f)$ and $f'_+(1)$ is not considered if $1 \in E(f)$.

Proof. First, if $E(f)$ contains infinite many points, then by compactness, there exists $x_0 \in E(f)$ which is also an accumulation point of $E(f)$. Since both $f'_-(x_0)$ (if $x_0 \neq 0$) and $f'_+(x_0)$ (if $x_0 \neq 1$) exist, at least one of them must be zero. Thus, by Corollary 3.3

$$\lim_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} = 0.$$

Now, we assume that $E(f)$ contains only finite points $x_1, x_2, ..., x_k$. For sufficient small $\alpha > 0$, a norming set of f can be expressed as

$$A_\alpha(f) = \bigcup_{i=1}^{k} [s_i, t_i],$$

and $|f(s_{i+1})| = |f(t_i)| = h_\alpha(f)$, $i = 1, ..., k-1$. Also, for sufficient small α,

$$|f(s_1)| = \begin{cases} \|f\| & \text{if} \quad s_1 = x_1 = 0 \\ h_\alpha(f) & \text{if} \quad x_1 > 0 \end{cases}$$

and

$$|f(t_k)| = \begin{cases} \|f\| & \text{if} \quad t_k = x_k = 1 \\ h_\alpha(f) & \text{if} \quad x_k < 1. \end{cases}$$

Then,

$$f'_-(x_i) + o(\alpha) = \frac{f(s_i) - f(x_i)}{s_i - x_i} = \operatorname{sgn}(f(x_i)) \frac{h_\alpha(f) - \|f\|}{s_i - x_i},$$

$$i = 2, 3, ..., k, \qquad \text{and} \qquad i = 1 \qquad \text{if} \quad x_1 \neq 0$$

and

$$f'_+(x_i) + o(\alpha) = \frac{f(t_i) - f(x_i)}{t_i - x_i} = \operatorname{sgn}(f(x_i)) \frac{h_\alpha(f) - \|f\|}{t_i - x_i},$$

$$i = 1, 2, ..., k-1, \qquad \text{and} \qquad i = k \qquad \text{if} \quad x_k \neq 1.$$

Solve for $x_i - s_i$ and $t_i - x_i$ from the above equalities and get

$$x_i - s_i = -\operatorname{sgn}(f(x_i)) \frac{h_\alpha(f) - \|f\|}{f'_-(x_i) + o(\alpha)} = \frac{\|f\| - h_\alpha(f)}{|f'_-(x_i)| + o(\alpha)},$$

$$t_i - x_i = -\operatorname{sgn}(f(x_i)) \frac{h_\alpha(f) - \|f\|}{f'_+(x_i) + o(\alpha)} = \frac{\|f\| - h_\alpha(f)}{|f'_+(x_i)| + o(\alpha)}$$

and

$$\begin{aligned}\alpha &= \sum_{i=1}^{k} [(x_i - s_i) + (t_i - x_i)] \\ &= (\|f\| - h_\alpha(f)) \sum_{i=1}^{k} \left(\frac{1}{|f'_-(x_i)| + o(\alpha)} + \frac{1}{|f'_+(x_i)| + o(\alpha)} \right).\end{aligned}$$

Then,

$$\begin{aligned}\frac{\|f\|_{(\alpha)} - \|f\|}{\alpha} &= \frac{1}{\alpha^2} \int_{A_\alpha(f)} |f(x)| - \|f\| \\
&= \frac{1}{\alpha^2} \sum_{i=1}^{k} \int_{s_i}^{t_i} (x - x_i) \operatorname{sgn}(f(x_i)) \frac{f(x) - f(x_i)}{x - x_i} \\
&= \frac{1}{\alpha^2} \sum_{i=1}^{k} \left[\int_{s_i}^{x_i} (x - x_i) \operatorname{sgn}(f(x_i)) \frac{f(x) - f(x_i)}{x - x_i} \right. \\
&\quad \left. + \int_{x_i}^{t_i} (x - x_i) \operatorname{sgn}(f(x_i)) \frac{f(x) - f(x_i)}{x - x_i} \right] \\
&= \frac{1}{\alpha^2} \sum_{i=1}^{k} \left[\operatorname{sgn}(f(x_i)) \frac{f(x_i^-) - f(x_i)}{x_i^- - x_i} \int_{s_i}^{x_i} (x - x_i) \right. \\
&\quad \left. + \operatorname{sgn}(f(x_i)) \frac{f(x_i^+) - f(x_i)}{x_i^+ - x_i} \int_{x_i}^{t_i} (x - x_i) \right] \\
&= -\frac{1}{2\alpha^2} \sum_{i=1}^{k} \left[\operatorname{sgn}(f(x_i)) \frac{f(x_i^-) - f(x_i)}{x_i^- - x_i} (x_i - s_i)^2 \right. \\
&\quad \left. + \operatorname{sgn}(f(x_i)) \frac{f(x_i^+) - f(x_i)}{x_i^+ - x_i} (t_i - x_i)^2 \right] \\
&= -\frac{1}{2} \sum_{i=1}^{k} \left[(|f'_-(x_i)| + o(\alpha)) \frac{(x_i - s_i)^2}{\alpha^2} \right. \\
&\quad \left. + (|f'_+(x_i)| + o(\alpha)) \frac{(t_i - x_i)^2}{\alpha^2} \right] \\
&= -\frac{1}{2(\sum_{i=1}^{k} (1/(f'_-(x_i) + o(\alpha)) + 1/(f'_+(x_i) + o(\alpha))))^2} \\
&\quad \times \sum_{i=1}^{k} \left[\frac{1}{|f'_-(x_i)| + o(\alpha)} + \frac{1}{|f'_+(x_i)| + o(\alpha)} \right] \\
&= -\frac{1}{2 \sum_{i=1}^{k} (1/(|f'_-(x)| + o(\alpha)) + 1/(|f'_+(x)| + o(\alpha)))},\end{aligned}$$

where $s_i < x_i^- < x_i$ and $x_i < x_i^+ < t_i$ whose existence follows from the Mean Value Theorem for the integral. Thus,

$$\lim_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} = -\frac{1}{2\sum_{x \in E(f)} (1/|f'_+(x)| + 1/|f'_-(x)|)}.$$

For the case that $\min_{x \in E(f)} \{|f'_+(x)|, |f'_-(x)|\} = \infty$, the proof is similar.

The following example shows that $\|f\|_{(\alpha)}$ may not be differentiable at $\alpha = 0$ if $f'_+(x)$ or $f'_-(x)$ does not exist for some $x \in E(f)$.

EXAMPLE 1. Let

$$f(x) = \begin{cases} 0, & 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{1}{4}\sum_{k=1}^{n-1} \frac{1}{2^k} + x - \sum_{k=1}^{n} \frac{1}{2^k}, & \sum_{k=1}^{n} \frac{1}{2^k} \leqslant x \leqslant \sum_{k=1}^{n} \frac{1}{2^k} + \frac{1}{2^{n+2}}, \quad n = 1, 2, \ldots \\ \frac{1}{4}\sum_{k=1}^{n} \frac{1}{2^k}, & \sum_{k=1}^{n} \frac{1}{2^k} + \frac{1}{2^{n+2}} \leqslant x \leqslant \sum_{k=1}^{n+1} \frac{1}{2^k}, \quad n = 1, 2, \ldots \end{cases}$$

$$\|f\| = \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{4} \qquad \text{and} \qquad f'_-(1) \text{ does not exist.}$$

If $\alpha = \sum_{k=n+1}^{\infty} (1/2^k) = 1/2^n$, then

$$\|f\|_\alpha = \frac{1}{1/2^n} \sum_{k=n}^{\infty} \left(\frac{1}{2^{k+1}} \frac{1}{4} \sum_{i=1}^{k-1} \frac{1}{2^i} + \frac{1}{2}\left(\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}}\right)\frac{1}{2^{k+2}} \right)$$

$$= 2^n \sum_{k=n}^{\infty} \left(\frac{1}{4} \frac{1}{2^{k+1}} \left(1 - \frac{1}{2^{k-1}}\right) + \frac{3}{8} \frac{1}{4^{k+1}} \right)$$

$$= 2^n \sum_{k=n}^{\infty} \left(\frac{1}{4} \frac{1}{2^{k+1}} - \frac{1}{8} \frac{1}{4^{k+1}} \right) = \frac{1}{4} - \frac{1}{8}\frac{1}{2^n}.$$

If $\alpha = \sum_{k=n+1}^{\infty} (1/2^k) - 1/2^{n+2} = 1/2^n - 1/2^{n+2}$, then

$$\begin{aligned}\|f\|_\alpha &= \frac{1}{1/2^n - 1/2^{n+2}} \left[\sum_{k=n}^{\infty} \left(\frac{1}{2^{k+1}} \frac{1}{4} \sum_{i=1}^{k-1} \frac{1}{2^i} + \frac{1}{2} \left(\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} \right) \frac{1}{2^{k+2}} \right) \right. \\ &\quad \left. - \frac{1}{2} \frac{1}{2^{n+2}} \left(\frac{1}{4} \sum_{k=1}^{n-1} \frac{1}{2^k} + \frac{1}{4} \sum_{k=1}^{n} \frac{1}{2^k} \right) \right] \\ &= \frac{2^{n+2}}{3} \left(\sum_{k=n}^{\infty} \left(\frac{1}{4} \frac{1}{2^{k+1}} - \frac{1}{8} \frac{1}{4^{k+1}} \right) - \frac{1}{16} \frac{1}{2^n} + \frac{3}{32} \frac{1}{4^n} \right) \\ &= \frac{1}{4} - \frac{1}{24} \frac{1}{2^n}.\end{aligned}$$

Thus, $\lim_{\alpha \to 0}((\|f\|_\alpha - \|f\|)/\alpha)$ does not exist.

THEOREM 3.5. *Let $f \in C[0, 1]$ and $0 < \alpha \leqslant 1$. Then*

$$\lim_{\beta \to \alpha^+} \frac{D_\beta(f) - D_\alpha(f)}{\beta - \alpha} = \frac{1}{\alpha} \Big(\inf_{p \in P_\alpha(f)} \{h_\alpha(f - p)\} - D_\alpha(f) \Big)$$

and

$$\lim_{\beta \to \alpha^-} \frac{D_\beta(f) - D_\alpha(f)}{\beta - \alpha} = \frac{1}{\alpha} \Big(\sup_{p \in P_\alpha(f)} \{h_\alpha(f - p)\} - D_\alpha(f) \Big).$$

If $h_\alpha(f - p)$ has the same value for all $p \in P_\alpha(f)$, then $D_\alpha(f)$ is differentiable with respect to α and

$$\frac{d}{d\alpha} D_\alpha(f) = \lim_{\beta \to \alpha} \frac{D_\beta(f) - D_\alpha(f)}{\beta - \alpha} = \frac{1}{\alpha} (h_\alpha(f - p) - D_\alpha(f)).$$

Proof. For $\alpha < \beta$, let $\bar{p}_\alpha \in P_\alpha$ such that $h_\alpha(f - \bar{p}_\alpha) = \inf_{p \in P_\alpha(f)} \{h_\alpha(f - p)\}$ and by Lemma 2.4 we can choose $p_\beta \in P_\beta$ such that

$$\lim_{\beta \to \alpha^+} h_\beta(f - p_\beta) = h_\alpha(f - \bar{p}_\alpha). \tag{4}$$

By Theorem 2.2, for each above p_β one can find a $p_{\alpha(\beta)} \in P_\alpha(f)$ such that

$$\lim_{\beta \to \alpha^+} \|p_{\alpha(\beta)} - p_\beta\| = 0, \tag{5}$$

and hence

$$\lim_{\beta \to \alpha^+} h_\beta(f - p_{\alpha(\beta)}) = h_\alpha(f - \bar{p}_\alpha). \tag{6}$$

Since $P_\alpha(f)$ is a compact set and all functions in this set are continuous, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $p_\alpha \in P_\alpha(f)$ and any x, $y \in [0, 1]$ with $|x - y| < \delta$

$$|p_\alpha(x) - p_\alpha(y)| < \varepsilon,$$

and then by (5)

$$|p_\beta(x) - p_\beta(y)| \leqslant |p_{\alpha(\beta)}(x) - p_{\alpha(\beta)}(y)| + 2 \, \|p_{\alpha(\beta)} - p_\beta\| < \varepsilon$$

for sufficient small δ and $0 \leqslant \beta - \alpha < \delta$.

Thus, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $x \in A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)$ with $|\alpha - \beta| < \delta$

$$\|f(x) - p_\beta(x)| - h_\beta(f - p_\beta)| < \varepsilon.$$

Also by (5)

$$\lim_{\beta \to \alpha^+} |h_\beta(f - p_{\alpha(\beta)}) - h_\beta(f - p_\beta)| = 0. \tag{7}$$

Then

$$\begin{aligned}
&\lim_{\beta \to \alpha^+} \sup_{x \in A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)} \|f(x) - p_{\alpha(\beta)}(x)| - h_\beta(f - p_{\alpha(\beta)})| \\
&\quad \leqslant \lim_{\beta \to \alpha^+} \sup_{x \in A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)} \|f(x) - p_\beta(x)| - h_\beta(f - p_\beta)| \\
&\qquad + \lim_{\beta \to \alpha^+} \|p_{\alpha(\beta)} - p_\beta\| + \lim_{\beta \to \alpha^+} |h_\beta(f - p_{\alpha(\beta)}) - h_\beta(f - p_\beta)| = 0.
\end{aligned} \tag{8}$$

Then, by (4), (5), (6), (7), and (8)

$$\begin{aligned}
&\int_{A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)} |f - p_\beta| \\
&\quad = (\beta - \alpha) \, h_\beta(f - p_\beta) + \int_{A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)} (|f - p_\beta| - |f - p_{\alpha(\beta)}|) \\
&\qquad + \int_{A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)} (|f - p_{\alpha(\beta)}| - h_\beta(f - p_{\alpha(\beta)})) \\
&\qquad + (\beta - \alpha)(h_\beta(f - p_{\alpha(\beta)}) - h_\beta(f - p_\beta)) \\
&\quad = (\beta - \alpha) \, h_\beta(f - p_\beta) + o(\beta - \alpha)
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&\int_{A_\beta(f-\bar{p}_\alpha)-A_\alpha(f-\bar{p}_\alpha)} |f-\bar{p}_\alpha| \\
&\qquad = (\beta-\alpha)\, h_\alpha(f-\bar{p}_\alpha) + \int_{A_\beta(f-\bar{p}_\alpha)-A_\alpha(f-\bar{p}_\alpha)} (|f-\bar{p}_\alpha| - h_\alpha(f-\bar{p}_\alpha)) \\
&\qquad = (\beta-\alpha)\, h_\alpha(f-\bar{p}_\alpha) + o(\beta-\alpha). \qquad (10)
\end{aligned}$$

Then, by (9) and (10)

$$\begin{aligned}
0 &\geqslant \beta(\|f-p_\beta\|_\beta - \|f-\bar{p}_\alpha\|_\beta) \\
&= \int_{A_\beta(f-p_\beta)} |f-p_\beta| - \int_{A_\beta(f-\bar{p}_\alpha)} |f-\bar{p}_\alpha| \\
&= \int_{A_\alpha(f-p_\beta)} |f-p_\beta| + \int_{A_\beta(f-p_\beta)-A_\alpha(f-p_\beta)} |f-p_\beta| \\
&\quad - \left[\int_{A_\alpha(f-\bar{p}_\alpha)} |f-\bar{p}_\alpha| + \int_{A_\beta(f-\bar{p}_\alpha)-A_\alpha(f-\bar{p}_\alpha)} |f-\bar{p}_\alpha|\right] \\
&= \int_{A_\alpha(f-p_\beta)} |f-p_\beta| + h_\beta(f-p_\beta)(\beta-\alpha) + o(\beta-\alpha) \\
&\quad - \left[\int_{A_\alpha(f-\bar{p}_\alpha)} |f-\bar{p}_\alpha| + h_\alpha(f-\bar{p}_\alpha)(\beta-\alpha) + o(\beta-\alpha)\right] \\
&= (h_\beta(f-p_\beta) - h_\alpha(f-\bar{p}_\alpha))(\beta-\alpha) + \alpha\, \|f-p_\beta\|_\alpha - \alpha D_\alpha(f) + o(\beta-\alpha) \\
&\geqslant (h_\beta(f-p_\beta) - h_\alpha(f-\bar{p}_\alpha))(\beta-\alpha) + o(\beta-\alpha).
\end{aligned}$$

From the above inequality and (4), we get

$$0 \geqslant \lim_{\beta\to\alpha^+} \frac{\|f-p_\beta\|_\beta - \|f-\bar{p}_\alpha\|_\beta}{\beta-\alpha} \geqslant \lim_{\beta\to\alpha^+} \frac{1}{\beta}\,(h_\beta(f-p_\beta) - h_\alpha(f-\bar{p}_\alpha)) = 0.$$

This shows

$$\lim_{\beta\to\alpha^+} \frac{\|f-p_\beta\|_\beta - \|f-\bar{p}_\alpha\|_\beta}{\beta-\alpha} = 0. \qquad (11)$$

Now,

$$D_\beta(f) - D_\alpha(f) = \|f-p_\beta\|_\beta - \|f-\bar{p}_\alpha\|_\beta + \|f-\bar{p}_\alpha\|_\beta - \|f-\bar{p}_\alpha\|_\alpha \qquad (12)$$

and by Theorem 3.1

$$\lim_{\beta \to \alpha^+} \frac{\|f-\bar{p}_\alpha\|_\beta - \|f-\bar{p}_\alpha\|_\alpha}{\beta-\alpha} = \frac{h_\alpha(f-\bar{p}_\alpha) - \|f-\bar{p}_\alpha\|_\alpha}{\alpha}$$

$$= \frac{h_\alpha(f-\bar{p}_\alpha) - D_\alpha(f)}{\alpha}. \tag{13}$$

Finally, combining (11), (12), and (13),

$$\lim_{\beta \to \alpha^+} \frac{D_\beta(f) - D_\alpha(f)}{\beta-\alpha} = \frac{h_\alpha(f-\bar{p}_\alpha) - D_\alpha(f)}{\alpha}$$

$$= \frac{1}{\alpha}\Big(\inf_{p \in P_\alpha(f)} \{h_\alpha(f-p)\} - D_\alpha(f)\Big).$$

Similarly we can prove

$$\lim_{\beta \to \alpha^-} \frac{D_\beta(f) - D_\alpha(f)}{\beta-\alpha} = \frac{1}{\alpha}\Big(\sup_{p \in P_\alpha(f)} \{h_\alpha(f-p)\} - D_\alpha(f)\Big).$$

The following example shows that $h_\alpha(f-p)$ may be different for different $p \in P_\alpha(f)$.

Example 2. Let

$$f(x) = \begin{cases} 1, & 0 \leqslant x \leqslant \frac{3}{4} \\ 4-4x, & \frac{3}{4} \leqslant x \leqslant 1, \end{cases} \qquad u(x) = \begin{cases} \frac{1}{4}x - \frac{1}{8}, & 0 \leqslant x \leqslant \frac{1}{2} \\ x - \frac{1}{2}, & \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

and $U = \operatorname{span}\{u\}$.

Then U is a unicity space of α-norm approximation for any α except $\alpha = \frac{3}{4}$ (see [6]). For $\alpha = \frac{3}{4}$, all $cu(x)$, $0 \leqslant c \leqslant 1$ are best α-norm approximation of f from U and $h_\alpha(f-cu)$ ranges from $\frac{3}{4}$ to 1.

Theorem 3.6. *Let $f \in C[0,1]$.*

(1) *If f and all $u \in U$ satisfy Lipschiz condition, i.e., there exists $L > 0$ such that for any $x, y \in [0,1]$*

$$|f(x) - f(y)| \leqslant L\,|x-y|$$

and for each $u \in U$, there exists $L_u > 0$ such that for any $x, y \in [0,1]$

$$|u(x) - u(y)| \leqslant L_u\,|x-y|,$$

then the exists $M>0$ *such that*

$$|D_\alpha(f)-D_0(f)|\leqslant M\alpha.$$

(2) *If* $P_0(f)=\{p_0\}$ *is singleton,* $u'(x)\in C[0,1]$ *for all* $u\in U$, *and* $f'(x)-p_0'(x)=0$ *for all* $x\in E(f-p_0)$, *then*

$$\lim_{\alpha\to 0}\frac{D_\alpha(f)-D_0(f)}{\alpha}=0.$$

Proof. Since $\bigcup_{0\leqslant\alpha\leqslant 1}P_\alpha(f)$ is a bounded set in a finite dimensional space and each of them satisfies Lipschiz condition, we can find a number $B>0$ such that for any $p\in\bigcup_{0\leqslant\alpha\leqslant 1}P_\alpha(f)$ and $|x-y|\leqslant\alpha$

$$|p(x)-p(y)|\leqslant B\alpha.$$

Let $M=L+B$ and then

$$\begin{aligned}|D_\alpha(f)-D_0(f)|&=D_0(f)-D_\alpha(f)\\&=\|f-p_0\|-\|f-p_\alpha\|+\|f-p_\alpha\|-\|f-p_\alpha\|_\alpha\\&\leqslant\|f-p_\alpha\|-\|f-p_\alpha\|_\alpha\leqslant\|f-p_\alpha\|-h(f-p_\alpha)\leqslant M\alpha.\end{aligned}$$

This proves the first part of the theorem. Now, for the second part, by Theorem 2.1 we have

$$0\leqslant\frac{D_0(f)-D_\alpha(f)}{\alpha}.$$

By Theorem 2.2, we can choose $p_\alpha\in P_\alpha(f)$ for each α so that $\lim_{\alpha\to 0}\|p_\alpha-p_0\|=0$. Since they are all from a finite dimensional space and have continuous derivatives, $\lim_{\alpha\to\infty}\|p'_\alpha-p'_0\|=0$. By a property of best uniform approximation there exists $x_0\in E(f-p_0)$ such that $\|f-p_0\|=|f(x_0)-p_0(x_0)|\leqslant|f(x_0)-p_\alpha(x_0)|$.

Then

$$\begin{aligned}&\frac{\|f-p_0\|-\|f-p_\alpha\|_\alpha}{\alpha}\\&\quad\leqslant\frac{1}{\alpha}\left[\,|f(x_0)-p_\alpha(x_0)|-\frac{1}{\alpha}\int_{A_\alpha(f-p_\alpha)}|f-p_\alpha|\right]\\&\quad\leqslant\frac{1}{\alpha^2}\int_{x_0}^{x_0+\alpha}(|(f(x_0)-p_\alpha(x_0))-(f(x)-p_\alpha(x))|)\end{aligned}$$

$$\leqslant \frac{1}{\alpha^2}\int_{x_0}^{x_0+\alpha}(|(f(x_0)-p_0(x_0))-(f(x)-p_0(x))|)$$

$$+\frac{1}{\alpha^2}\int_{x_0}^{x_0+\alpha}(|p_\alpha(x_0)-p_0(x_0)|-|p_\alpha(x)-p_0(x)|.$$

For small α, the first term

$$\left|\frac{1}{\alpha^2}\int_{x_0}^{x_0+\alpha}(|(f(x_0)-p_0(x_0))-(f(x)-p_0(x))|)\right|$$

$$\leqslant \max_{x_0\leqslant x\leqslant x_0+\alpha}\left|\frac{(f(x)-p_\alpha(x))-(f(x_0)-p_\alpha(x_0))}{\alpha}\right|.$$

It goes to 0 by the assumption that the derivatives of $f-p_\alpha$ are all zero at any $x\in E(f)$. The second term

$$\frac{1}{\alpha^2}\int_{x_0}^{x_0+\alpha}(|(p_\alpha(x_0)-p_0(x_0))-(p_\alpha(x)-p_0(x))|)$$

$$\leqslant \max_{x_0\leqslant x\leqslant x_0+\alpha}\left|\frac{(p_\alpha(x)-p_0(x))-(p_\alpha(x_0)-p_0(x_0))}{\alpha}\right|$$

also goes to 0 because $\lim_{\alpha\to\infty}\|p'_\alpha-p'_0\|=0$.

The second part of the theorem is proved.

Comparing part (2) of Theorem 3.6 and part (2) of Corollary 3.3, one might ask if the condition $f'(x)-p'_0(x)=0$ for all $x\in E(f-p_0)$ can be replaced by $f'(x)-p'_0(x)=0$ for one $x\in E(f-p_0)$. the following example shows it cannot

EXAMPLE 3. Let

$$f(x)=\begin{cases}1-2x, & 0\leqslant x\leqslant \frac{1}{2}\\ (2x-1)(2x-3), & \frac{1}{2}\leqslant x\leqslant 1,\end{cases}$$

$$u(x)=\begin{cases}1, & 0\leqslant x\leqslant \frac{2}{5}\\ \text{linear}, & \frac{2}{5}\leqslant x\leqslant \frac{3}{5}\\ 3, & \frac{3}{5}\leqslant x\leqslant 1,\end{cases}$$

and $U = \text{span}\{u\}$. Then $p_0(f) = 0$, $D_0(f) = \|f\| = 1$, and $f'(1) - p_0'(f)(1) = 0$, but, for $\alpha \leqslant \frac{1}{4}$,

$$\frac{D_\alpha(f) - D_0(f)}{\alpha} \leqslant \frac{\left\|f + \frac{1}{2}\alpha u\right\|_\alpha - \|f\|}{\alpha}$$

$$= \frac{1}{\alpha}\left(\frac{1}{\alpha}\int_0^\alpha \left(1 - 2x + \frac{1}{2}\alpha\right) dx - 1\right)$$

$$= -\frac{1}{2} < 0.$$

THEOREM 3.7. *Let $f \in C[0, 1]$ and U be a Chebyshev space.*

(1) *If f and all $u \in U$ satisfy Lipschiz condition, then for any $p_\alpha \in P_\alpha(f)$,*

$$\|p_\alpha - p_0\| \leqslant C\alpha, \qquad \textit{for some constant } C.$$

(2) *If $f'(x), u'(x) \in C[0, 1]$ for all $u \in U$ and $f'(x) - p_0'(x) = 0$ for all $x \in E(f - p_0)$, then*

$$\lim_{\alpha \to 0} \frac{\|p_\alpha - p_0\|}{\alpha} = 0.$$

Proof. By the same reason given in the proof of Theorem 3.6, there exists $L > 0$ such that for any $|x - y| \leqslant \alpha$ and any $p \in \bigcup_{0 \leqslant \alpha \leqslant 1} P_\alpha(f)$

$$|f(x) - f(y)| \leqslant \alpha \qquad \text{and} \qquad |p(x) - p(y)| \leqslant \alpha.$$

By the strong uniqueness of the best uniform approximation, there exists $\gamma = \gamma(f) > 0$ such that

$$\|f - u\| \geqslant \|f - p_0\| + \gamma\, \|p_0 - u\|$$

for any $u \in C[0, 1]$. Also, $h(f - p_\alpha) \leqslant \|f - p_\alpha\|_\alpha \leqslant \|f - p_0\|_\alpha \leqslant \|f - p_0\|$.

Now, replace u by p_α and get

$$\|p_0 - p_\alpha\| \leqslant \frac{1}{\gamma}(\|f - p_\alpha\| - \|f - p_0\|)$$

$$\leqslant \frac{1}{\gamma}(\|f - p_\alpha\| - h(f - p_\alpha)) \leqslant \frac{2L}{\gamma}\alpha.$$

For the second part of the theorem, we have

$$
\begin{aligned}
0 &\leqslant \lim_{\alpha \to 0} \frac{\|p_\alpha - p_0\|}{\alpha} \leqslant \lim_{\alpha \to 0} \frac{1}{\gamma}\left(\frac{\|f - p_\alpha\| - \|f - p_0\|}{\alpha}\right) \\
&\leqslant \frac{1}{\gamma} \lim_{\alpha \to 0} \frac{\|f - p_\alpha\| - \|f - p_\alpha\|_\alpha}{\alpha} \\
&\leqslant \lim_{\alpha \to 0} \frac{1}{\alpha} \int_{x_\alpha}^{x_\alpha + \alpha} \left(\frac{|(f - p_\alpha)(x_\alpha)| - |(f - p_\alpha)(x)|}{\alpha}\right) \\
&\leqslant \max_{x_\alpha \leqslant x \leqslant x_\alpha + \alpha} \frac{|(f - p_\alpha)(x_\alpha)| - |(f - p_\alpha)(x)|}{\alpha} = 0.
\end{aligned}
$$

where $|(f - p_\alpha)(x_\alpha)| = \|f - p_\alpha\|$.

This proves the second part of the theorem.

REFERENCES

1. E. W. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
2. A. G. Egger and G. D. Taylor, Dependence on p of the best L^p approximation operator, *J. Approx Theory* **46** (2) (1983), 88–95.
3. B. R. Kipke and T. J. Rivlin, Approximation in the metric of $L^1(X, \mu)$, *Amer. Math. Soc.* **119** (1965), 101–112.
4. E. Lapidot and J. T. Lewis, Best approximation using a peak norm, *J. Approx Theory* **67** (2) (1991), 174–186.
5. C. Li and G. A. Watson, On approximation using a peak norm, *J. Approx Theory* **77** (3) (1994), 266–275.
6. C. Yang, Uniqueness of best α-norm approximation and A-spaces, *Numer. Funct. Anal. Optim.* **13** (1992), 403–412.